The Exact Multi-User Security of Key-Alternating Feistel Ciphers with a Single Permutation

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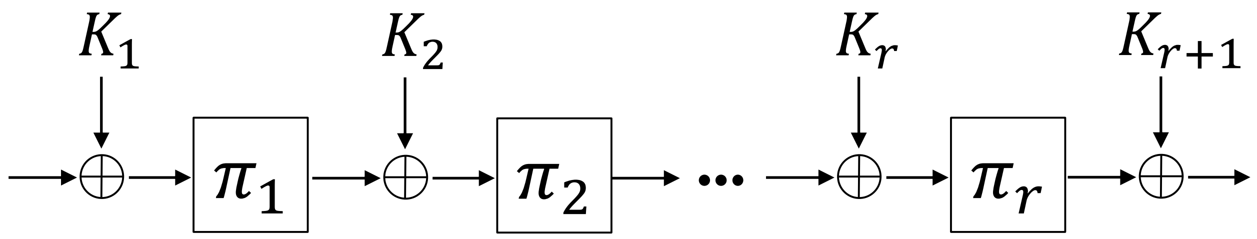
Security of Generic Block Cipher Construction

- It is popular to generalize constructions and study their security.

  ➡ The results are applied to many designs in general.

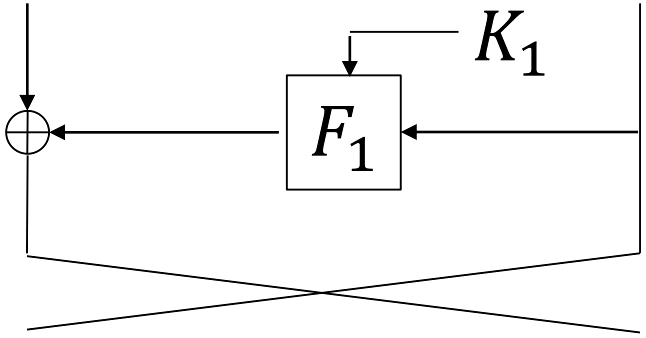
- The goal is to drive the lower and upper bounds of the construction to be distinguished from ideal n-bit SPRP.

Key Alternating Ciphers (KACs)



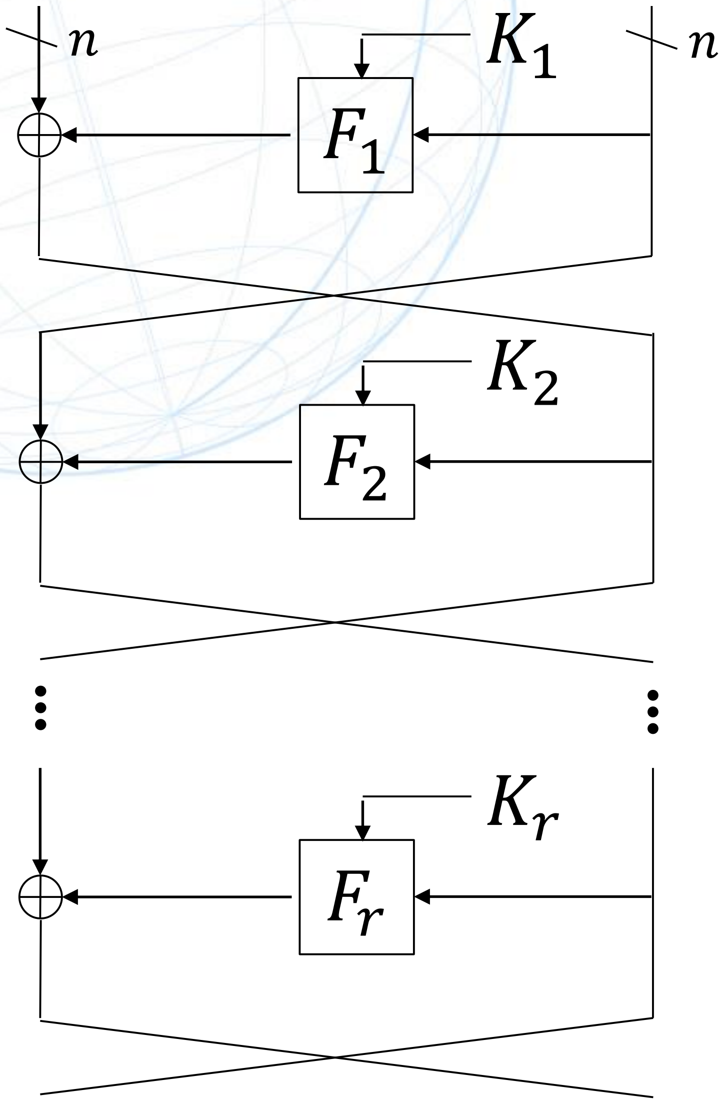
Studied at Eurocrypt 2024 by Naito-Sasaki-Sugawara

Feistel Ciphers



This paper !!

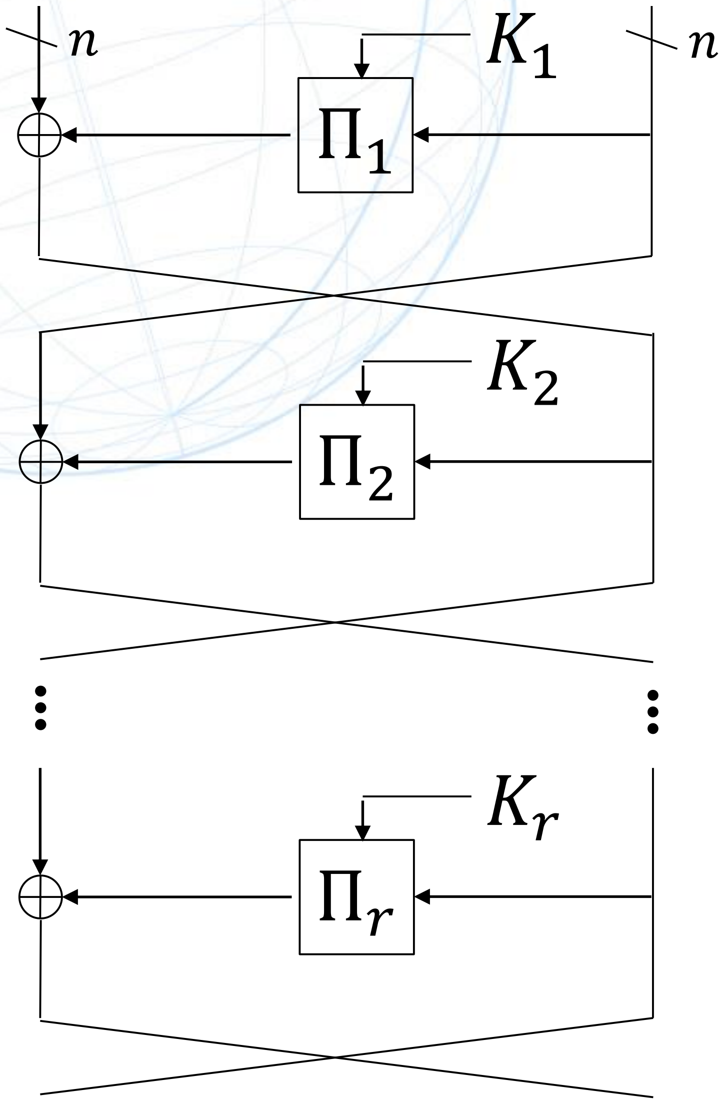
Luby-Rackoff



- It was proposed by Luby and Rackoff in 1986.
- The size of each branch is n bits.
- Round functions are secret and independent in each round.

- Patarin proved that 4 rounds are SPRP up to $O(2^{\frac{1}{2}n})$ queries.
- Many other results are known …

Luby-Rackoff with Pemutation



- First analyzed by Piret in 2006.
- Motivated by the fact that practical designs mostly adopt permutations as round functions.
- This direction was subsequently continued by Guo and Zhang [17] in 2021.

KAF-F: Feistel with Key Alternating Function

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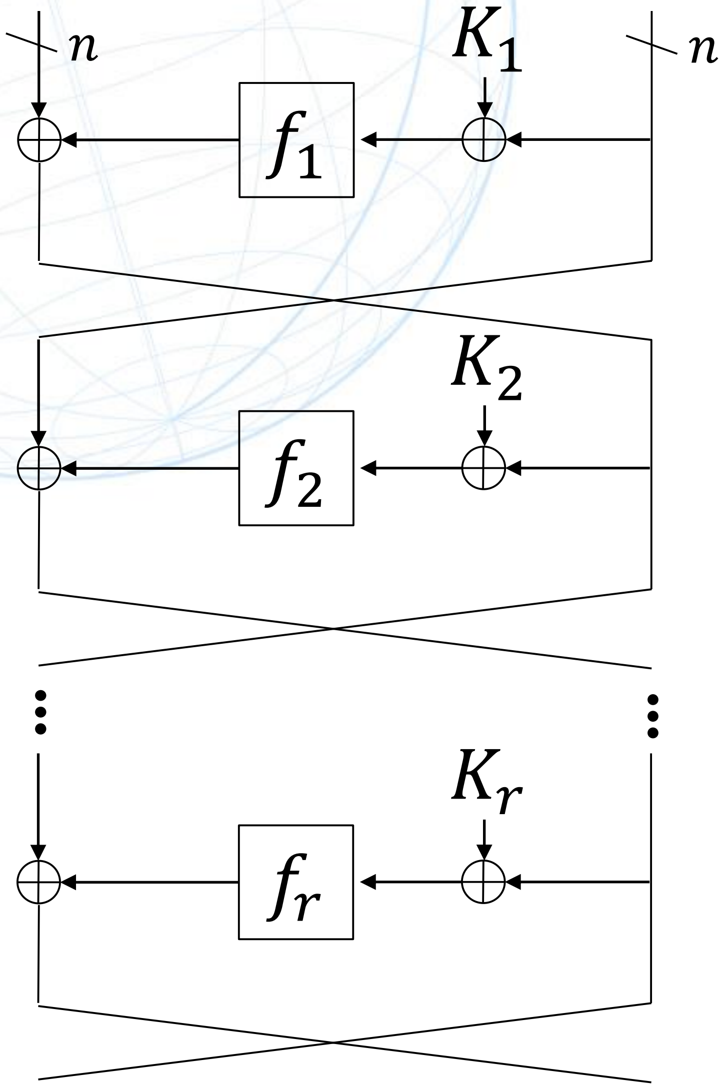
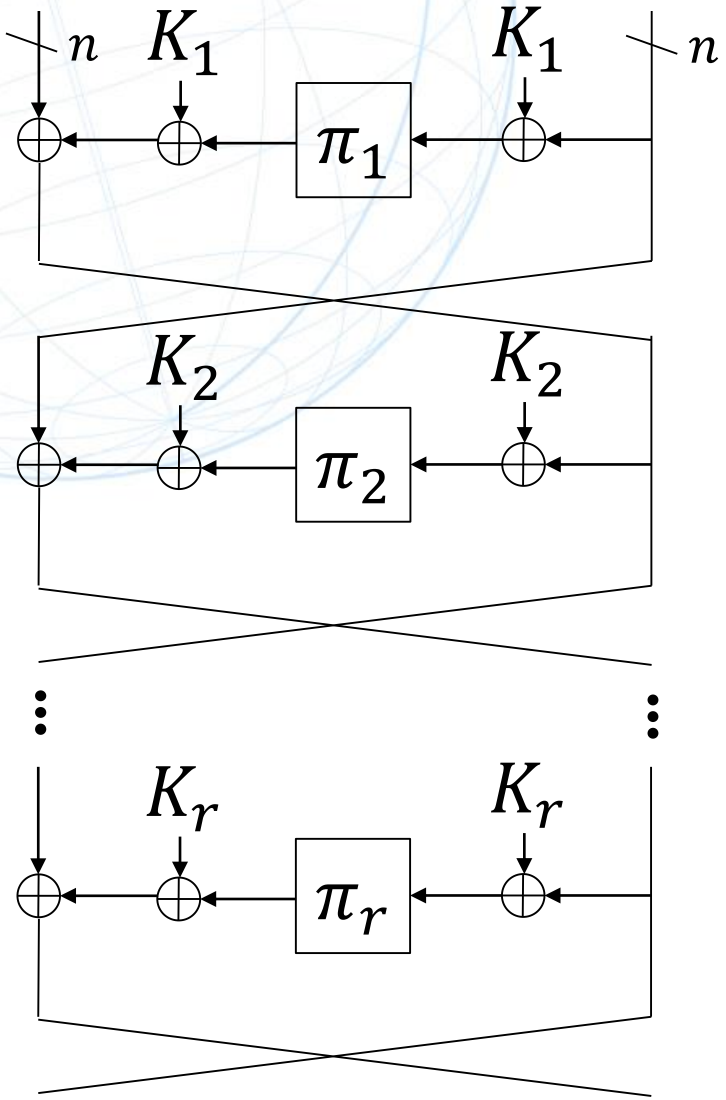


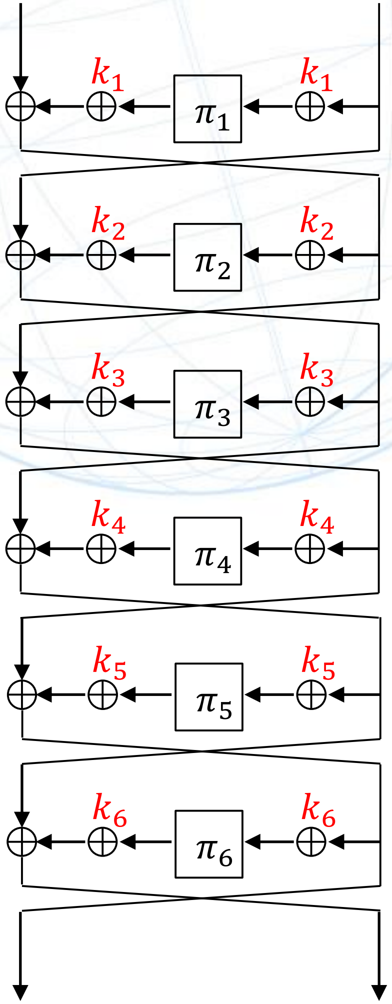
- Studied by Lampe-Seurin in 2014.
- Motivated by the fact that practical designs mostly adopt round functions applying a key and a public function.
- Big change in security analysis since adversaries now can make primitive queries besides construction queries.
- [LS14] proved that $6t$ rounds are SPRP up to $O(2^{\frac{t}{t+1}n})$ queries.
- Guo-Wang [GW18] proved that
  - 4-rounds with 1 key: $O(2^{\frac{n}{2}})$
  - 6-rounds with 2 key: $O(2^{\frac{2n}{3}})$

KAF-P: Feistel with Even-Mansour

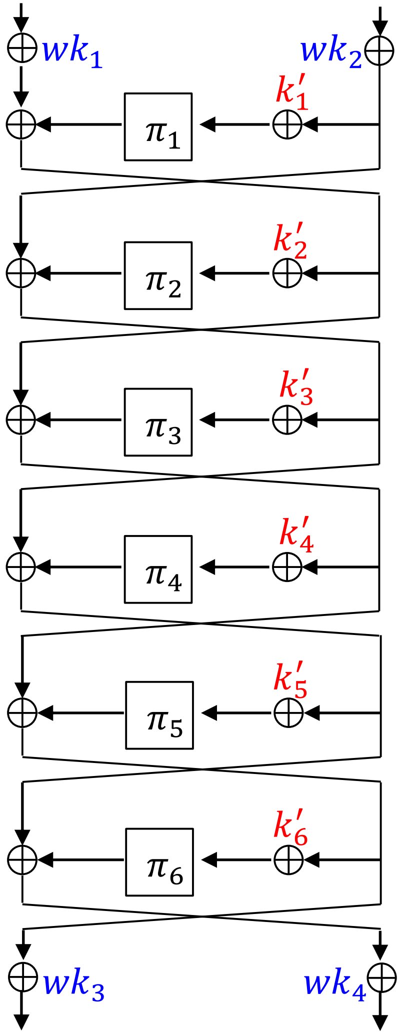


- First studied by Bhattacharjee et al. in 2024.
- Motivated by the fact that practical designs mostly adopt a public permutation.
- It was proved that 5 rounds are SPRP up to $O(2^{\frac{2}{3}n})$ queries.
- We further show that if KAF-P is secure, so is **whitening + key + π**.

KAF-P is Secure ⇒ Practical Designs are Secure

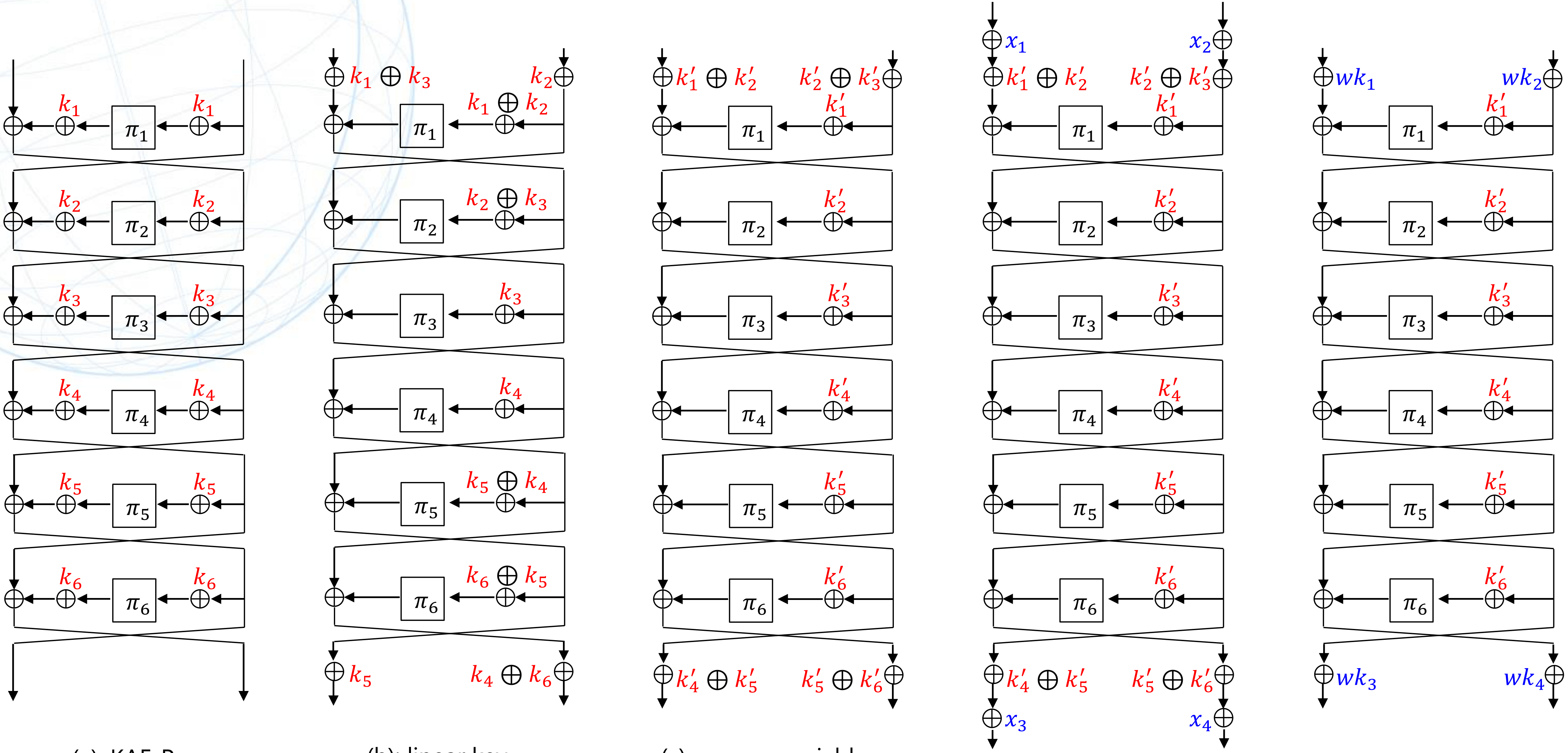


(a): KAF-P



(e): practical structure

KAF-P is Secure ⇒ Practical Designs are Secure



(a): KAF-P

(b): linear key transformation

(c): rename variables

(d): add 4 keys x_1, x_2, x_3, x_4 to strengthen the scheme

(e): practical structure

Research Directions

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- **Tightness**: generic attacks matching the proven upper bound should be provided.
- **Multi-user security**: Adversaries make queries to multiple users having independently generated keys. This model captures more realistic cases.
- **Single-primitive**: Proofs are simpler if primitives are independently chosen in every round, while practical designs usually use only a single primitive for efficiency.
- **Correlated Subkeys**: Proofs are simpler if all the subkeys are independent, while practical designs usually generate all the subkeys from a master key.

Comparison of Results

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- We prove that r rounds of KAF-P is secure up to $O(2^{\frac{r-2}{r-1}n})$ queries.
  tight, multi-user, single primitive, $r-2$ independent keys

Table 1. Provable security bounds of Feistel ciphers with public primitives.

Reference	Type	Round	Bound (bits)	Tightness	Model	Single Primitive	Indep. Subkeys†
Lampe–Seurin [26]	KAF-F	12	$\frac{2}{3}n$	—	su	—	All
Lampe–Seurin [26]	KAF-F	$6t$	$\frac{t}{t+1}n$	—	su	—	All
Guo–Wang [16]	KAF-F	4	$\frac{1}{2}n$	✓	mu	✓	1
Guo–Wang [16]	KAF-F	6	$\frac{2}{3}n$	—	mu	—	2
Bhattacharjee et al. [4]	KAF-P	5	$\frac{2}{3}n$	—	su	—	All
Ours	KAF-P†	r	$\frac{r-2}{r-1}n$	✓	mu	✓	$r-2$

†Our attack is also applicable to KAF-F.

Best Generic Attacks for 5 Rounds

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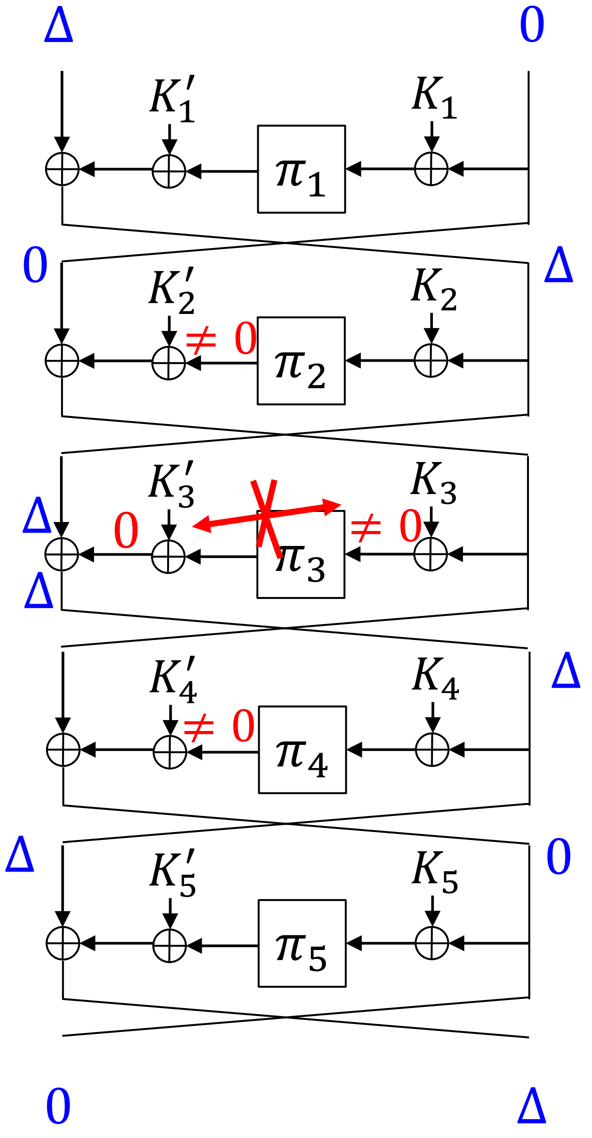
Impossible Differential Attacks

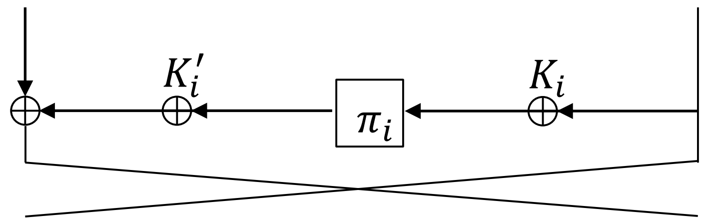
- The difference $(\Delta, 0)$ never propagates to difference $(0, \Delta)$ after 5 rounds.
- This property allows to distinguish 5 rounds with $O(2^n)$ queries.
- This type of attacks will be inapplicable when r becomes large, since any differential propagation will be possible for a large r.



Target Constructions in our Attacks / Proofs

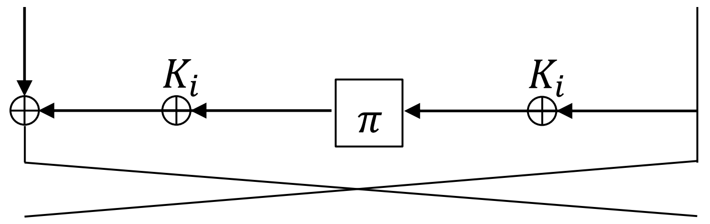
Attacks

- Attacks are better if it works even if all rounds use independent permutation and independent subkeys, moreover different keys for Even-Mansour construction.



Proofs

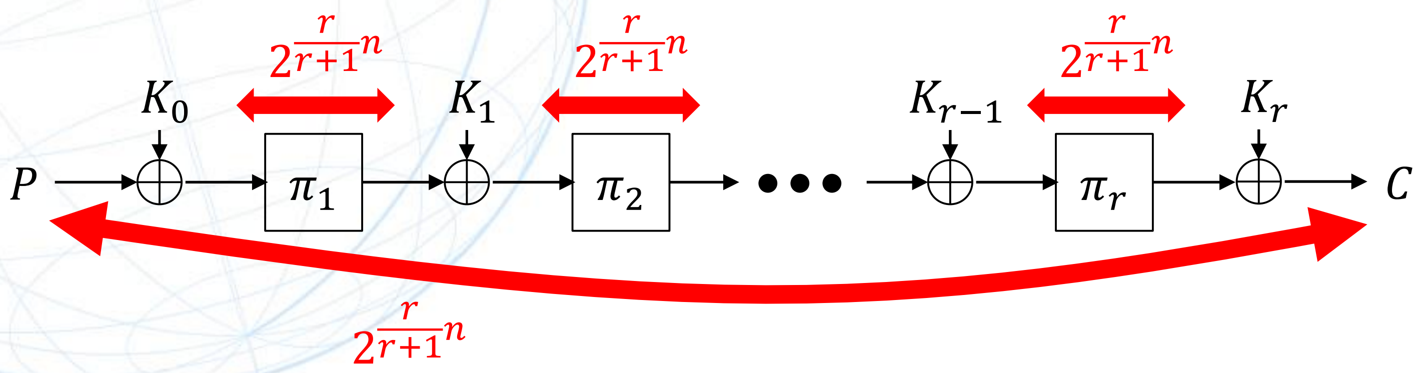
- Proofs are better if it works even if all rounds use the same permutation and the same key for the Even-Mansour construction.



New Attacks

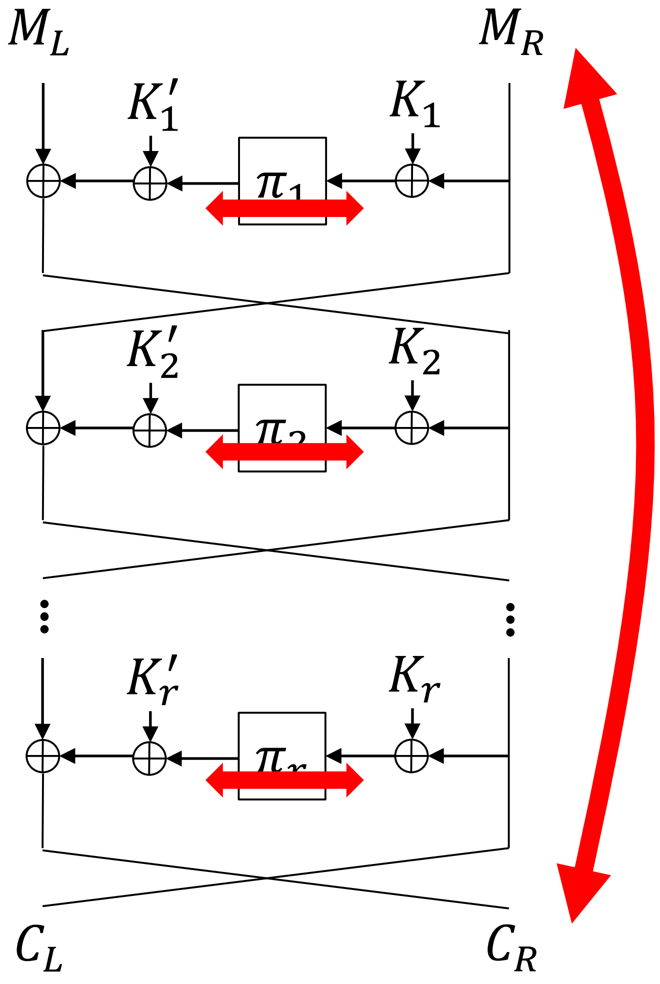
Inapplicability of Related Works 1

Generic Attacks on r-round KAC



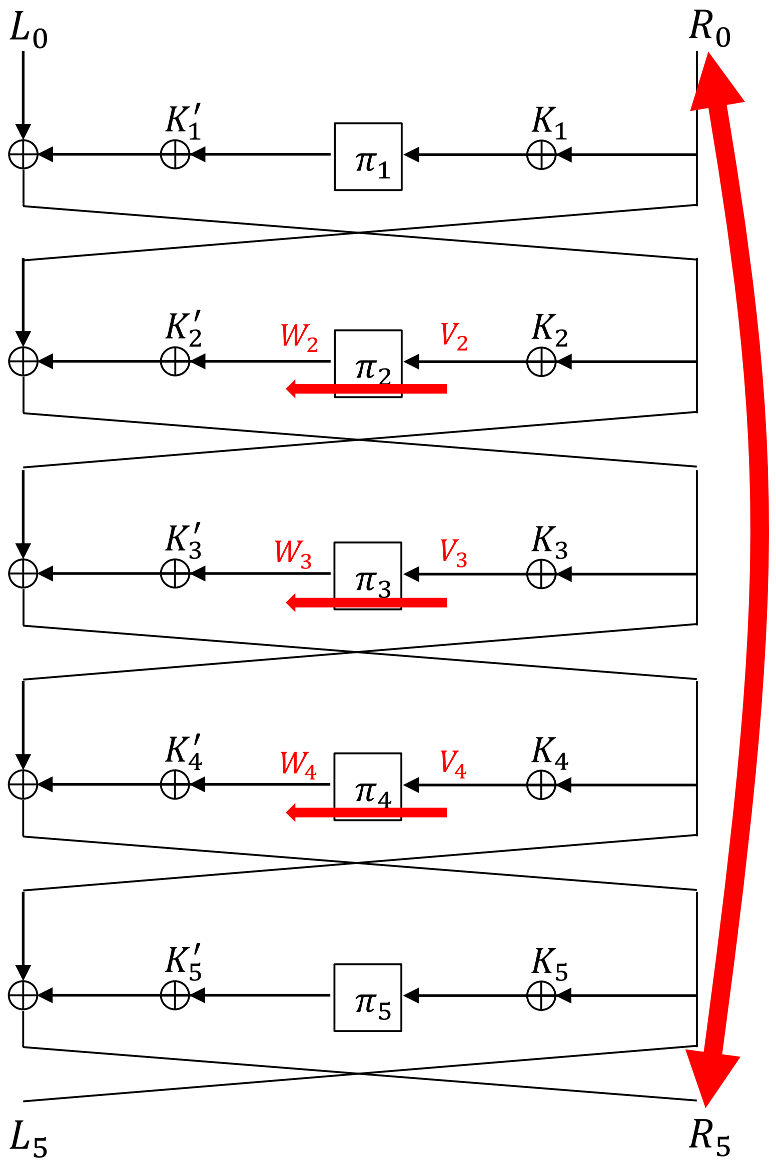
- Make $O(2^{\frac{r}{r+1}n})$ construction queries.
- Make $O(2^{\frac{r}{r+1}n})$ primitive queries for each π_i.
- There should exist consistent queries.
- Subkeys are derived just computing XORs.

However, for Feistel, even if both queries match, XOR of Feistel construction protects subkeys.



Our Approach: Meet-in-the-Middle

- We first find a match between construction and primitive queries for all but the first and the last rounds; i.e. a consistent tuple $L_0||R_0, (V_2, W_2), (V_3, W_3), \ldots, (V_{r-1}, W_{r-1}), L_r||R_r$
- To recover subkeys, we make it a pair with another construction query, and to trace differential propagation rather than values. (propagate with prob.1 over subkey XOR)
- Values after π_i for the query that is chosen to be a pair can be looked up by reusing primitive queries.



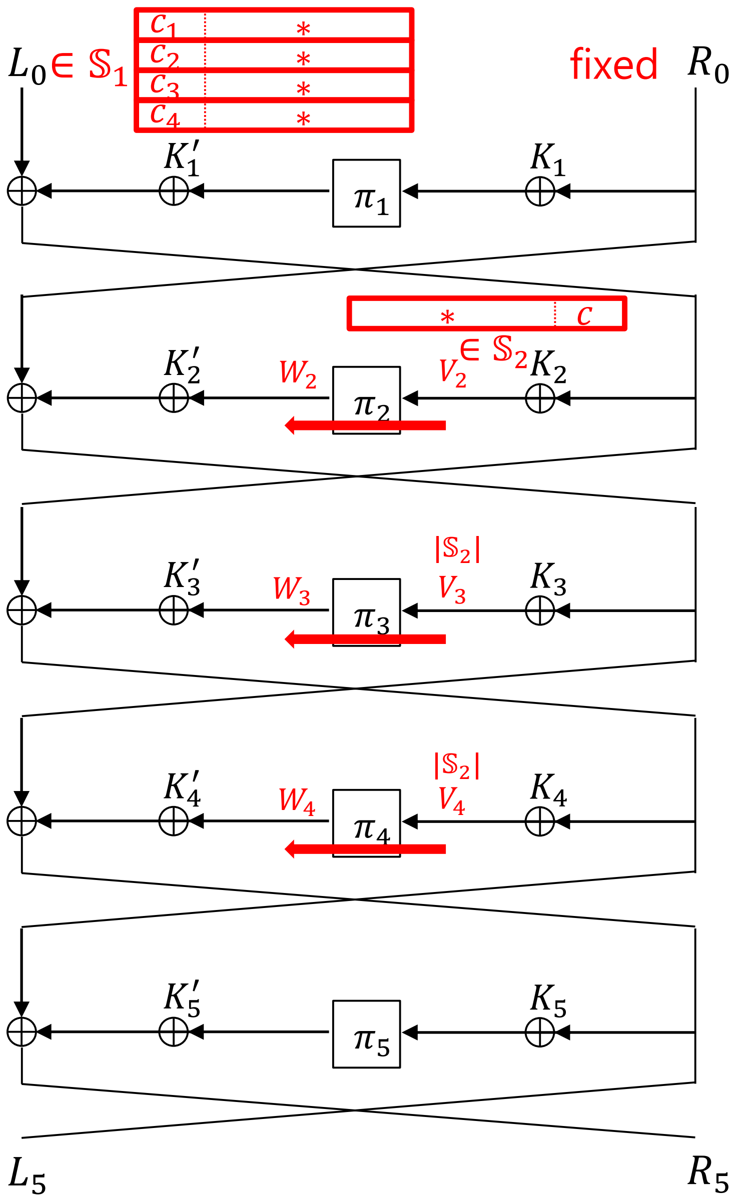
Figures are for 5 rounds.

Query Strategy

- Definition of Set $\mathbb{S}_1$:
  - MSB: $n - \frac{r-2}{r-1}n$ bits are constant (c_i)
  - LSB: $\frac{r-2}{r-1}n$ bits take all values
- Definition of Set $\mathbb{S}_2$:
  - MSB: $\frac{r-2}{r-1}n$ bits take all values
  - LSB: $n - \frac{r-2}{r-1}n$ bits are constant (c)

- Construction Queries
  - Query $r-2$ sets of $\mathbb{S}_1$
- Primitive Queries
  - Query $\mathbb{S}_2$ for all but the first and the last rounds.

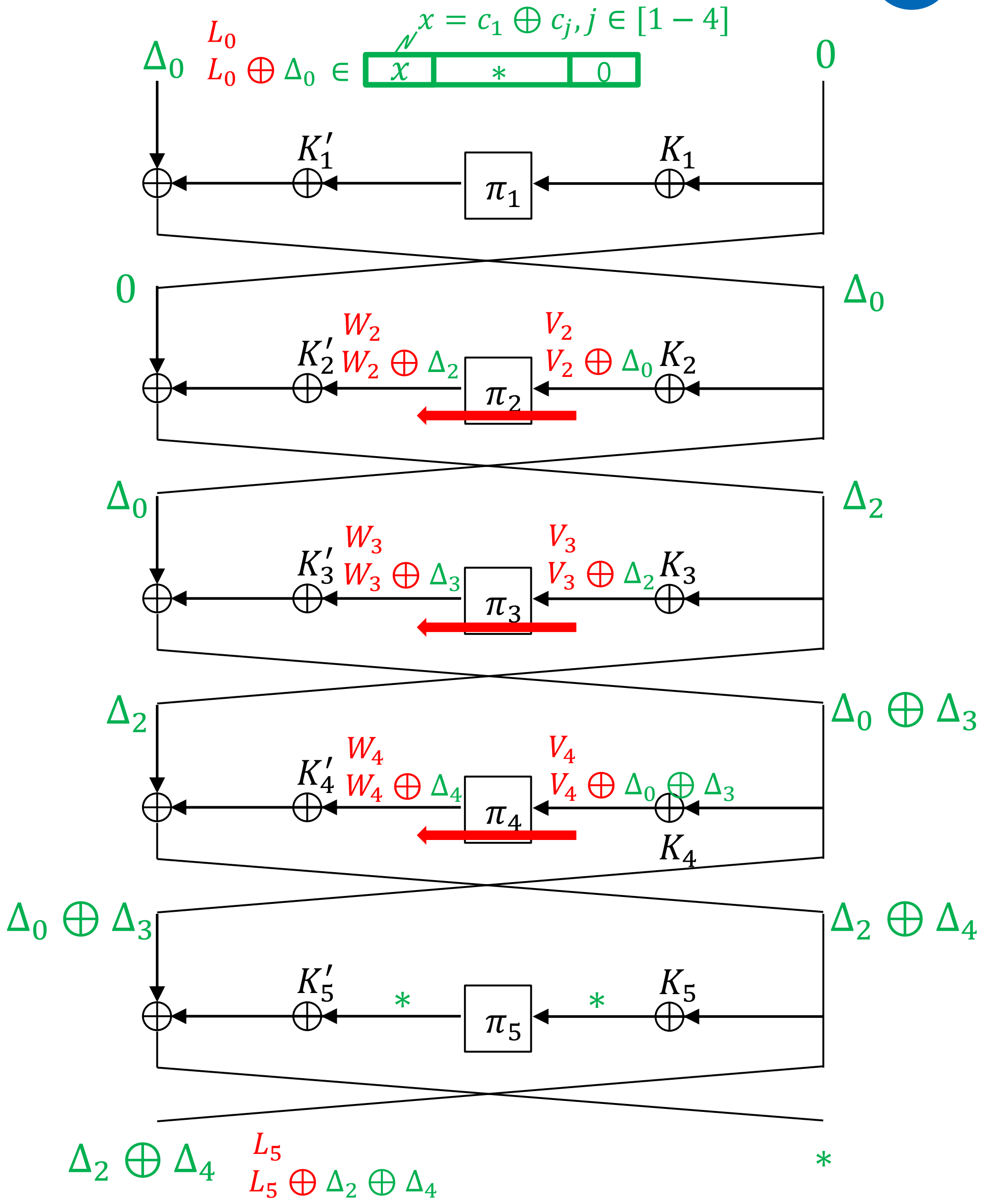
By taking any combination of construction and primitive queries, a match is expected.



Distinguished Procedure

- For all $L_0||R_0, (V_2, W_2), \ldots, (V_{r-1}, W_{r-1}), L_r||R_r$, make a pair with $L'_0||R'_0, L'_r||R'_r$.

1. 1st Round: Δ_0 is simply computed.
2. 2nd Round: V'_2 is computed $V_2 \oplus \Delta_0$. V'_2 exists in primitive queries, so it's possible to look up W'_2. Then, $\Delta_2 = \mathrm{W}_2 \oplus W'_2$ can be computed.
3. 3rd to r-1 rounds: V'_i is computed $V_i \oplus \Delta_{i-1}$. If V'_i exists in primitive queries, then look up W'_i and compute $\Delta_i = \mathrm{W}_i \oplus W'_i$.
4. Last round: Check the correctness of the pair by matching the left-half of the ciphertext.

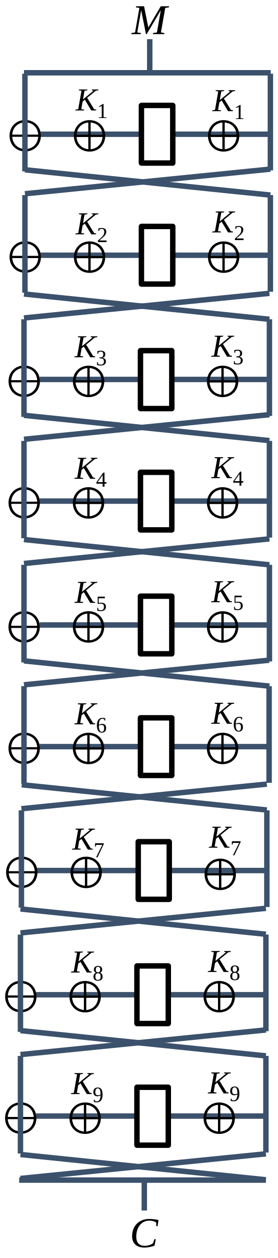


Green represents differences.

New Proofs

Proof with Resampling Method

- Tight mu-bound: $\frac{r-2}{r-1}n$ bits
  for KAF-P with a single permutation.
- Proof Methods:
  - Patarin's coefficient-H technique.
  - Resampling method with new procedures for KAF-P.
- Resampling method for any r
  - Introduced for Key Alternating Cipher at EUROCRYPT2024.
  - Define dummy internal values for each (M,C)
    by forward and backward sampling steps in the ideal word.
    1. Perform a forward sampling.
    2. Perform an inverse sampling
       if a collision occurs for some internal value.



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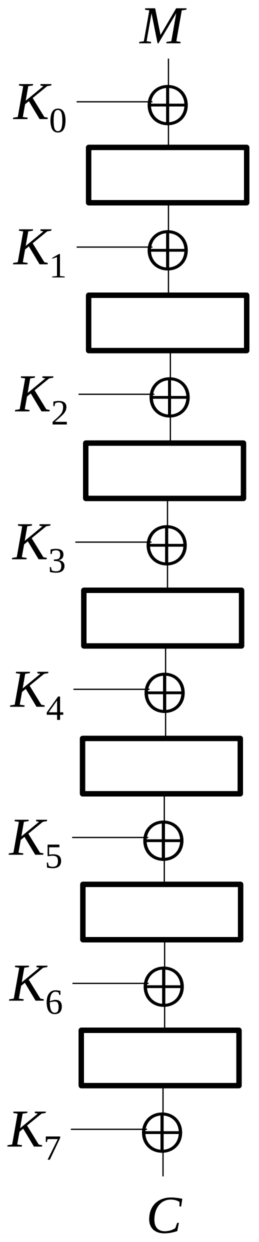

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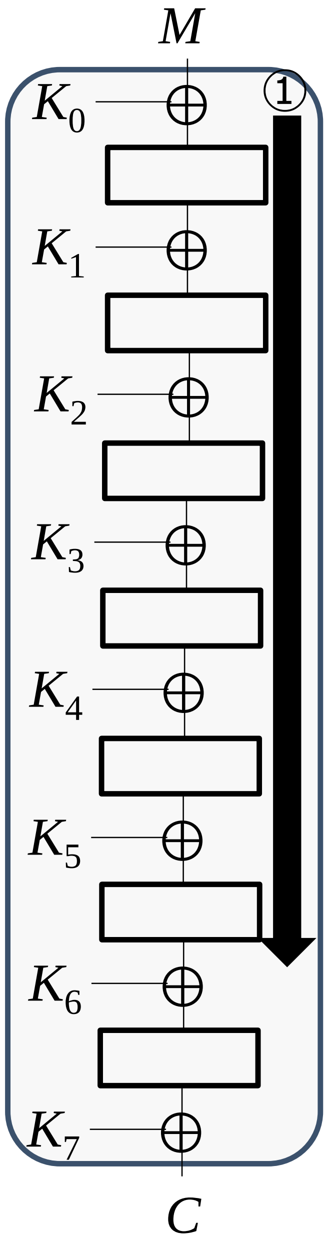

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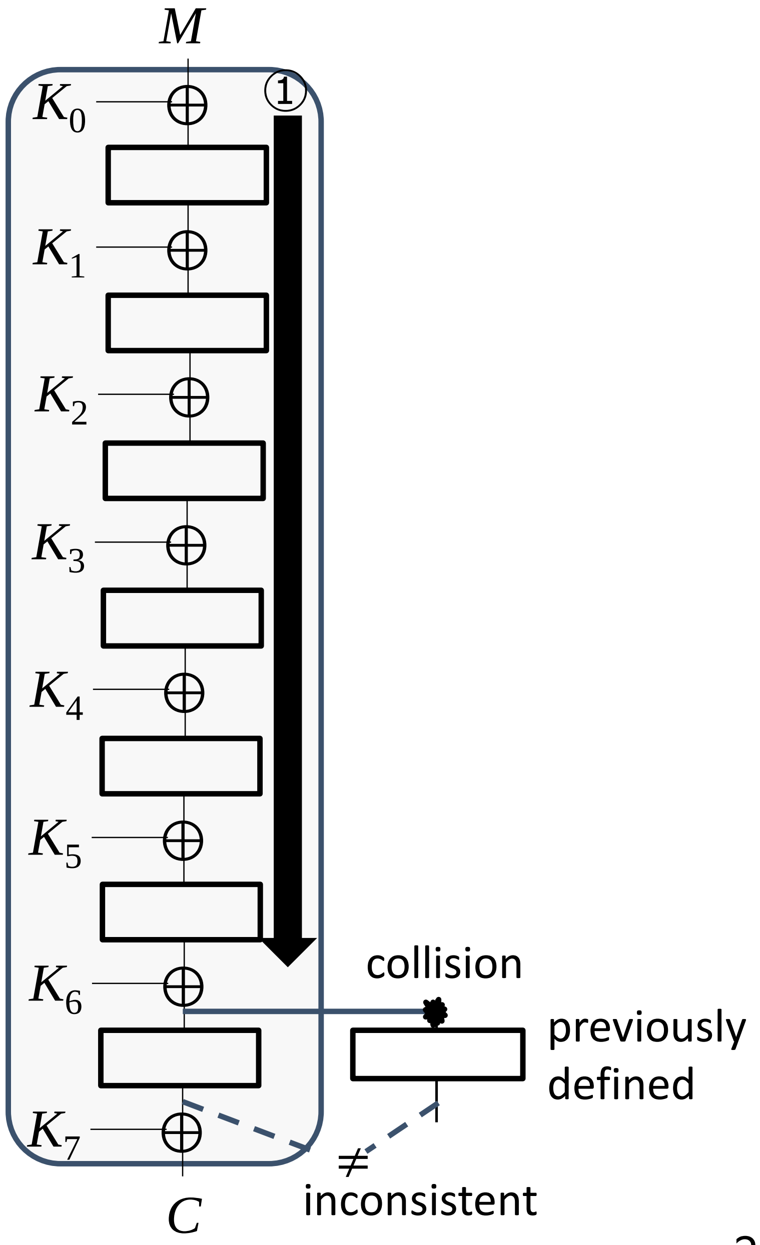

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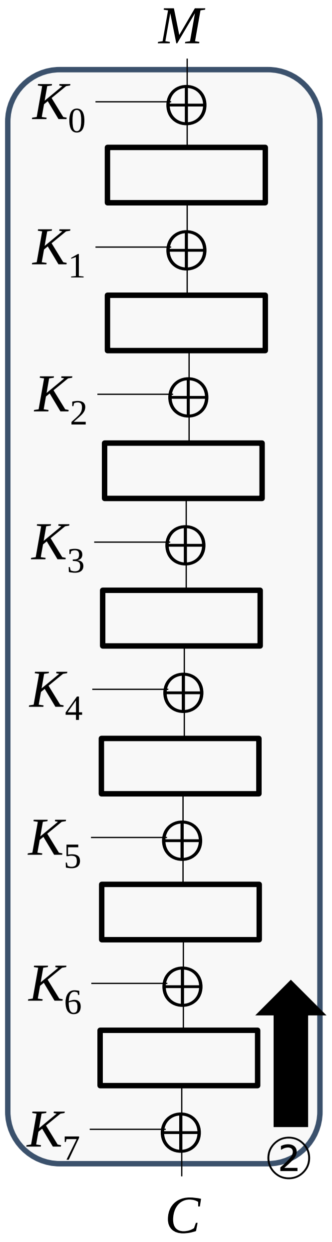
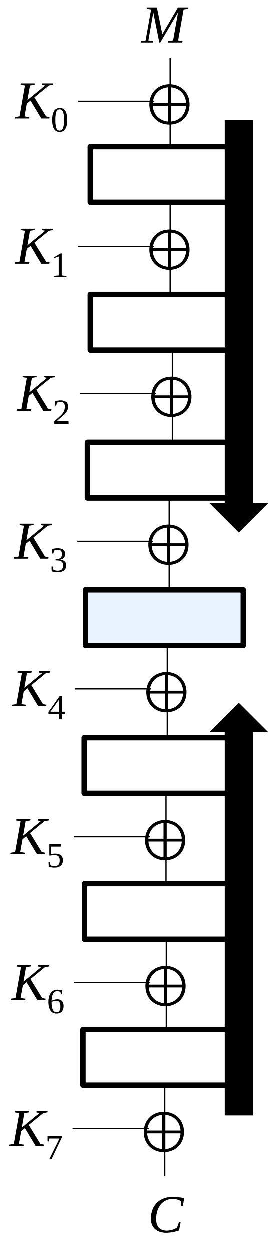
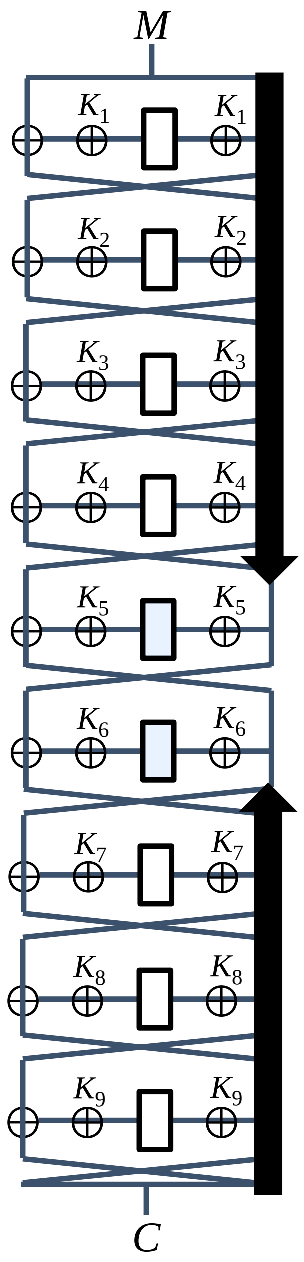


Resampling Method for KAF−P

- Update the resampling method for KAF-P with a single permutation.
- Differences between KAC and KAF−P.
  - KAC: $r-1$ internal values define all internal values.
  - KAF−P: $r-2$ internal values define all internal values.
- Collision events for failures of the resampling method.
  - KAC: 1
  - KAF−P: 3
- We give a new resampling algorithm for KAF−P with the three collision events
  ⇒ Tight mu-bound for KAF-P: $\frac{r-2}{r-1}n$ bits.





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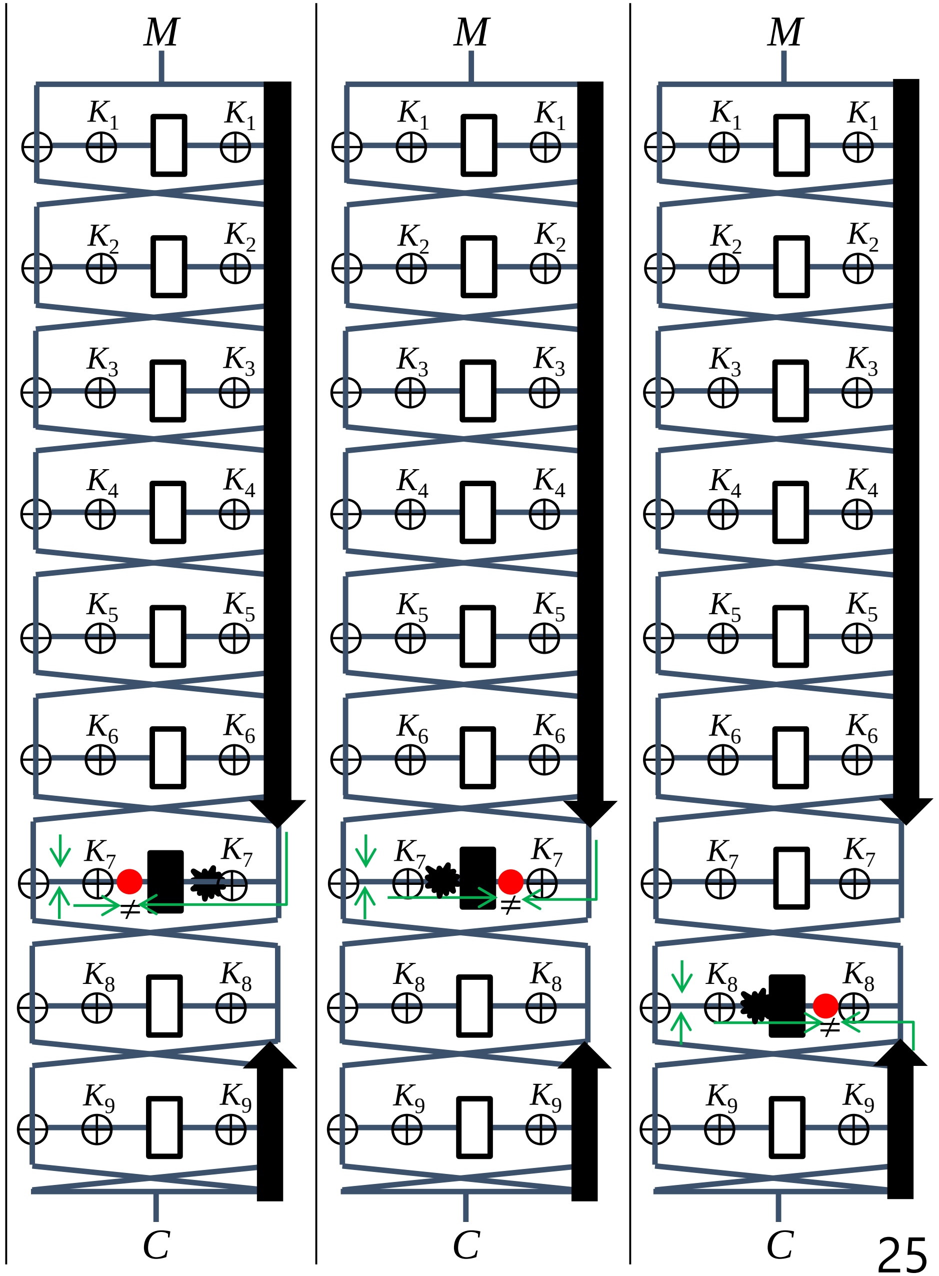
  $\Rightarrow$ Tight mu-bound for KAF-P: $\frac{r-2}{r-1}n$ bits.



Conclusion

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- Provable tight security bound of Feistel KAF-P ciphers
  - in the multi-user (mu) setting
  - a single primitive across all rounds
  - $r-2$ correlated subkeys for r rounds
- By applying the resampling method to Feistel KAF-P ciphers, security is proven to be $O(2^{\frac{r-2}{r-1}n})$ for r rounds.
- We also provide a new matching attack by information-theoretic variant of the meet-in-the-middle attack.

Thank you for your attention!!